Effect of Lagging Pitching Moment on Re-Entry Vehicle Dynamic Stability

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Recent work has shown that the effect of ablation on re-entry vehicle (R/V) dynamic stability may be significant. Waterfall has explained flight test anomalies by postulating a dynamic model in terms of an ablative pitching moment derivative $C_{m_{a_\alpha}}$ and associated time lag τ and by assuming that the resulting behavior was representable in the classical two-frequency form. The problem has been treated from a similar point of view by Ericsson. The present work is concerned with establishing the conditions under which the two-frequency assumption is valid and examing the effects of large phase lags. Exact solutions of the difference-differential equation which results from the inclusion of the term with retarded argument have been generated for the special case of constant dynamic pressure for a range of pertinent aerodynamic parameters. A significant result is that even for large time lags ($\tau \sim 1$ sec) the solution retains its classical two-component form. The solutions are compared to the approximate technique of Waterfall and a convenient means of generating the solutions is presented.

Nomenclature

A, B = roll, pitch moments of inertia
C_{m_α} = pitching moment coefficient
$C_{m_{a_\alpha}}$ = ablation induced pitching moment derivative
C_{m_q} = dynamic stability derivative
$C_{n_{p_\alpha}}$ = Magnus moment derivative
C_{N_α} = normal force coefficient derivative
d = body reference length
e = classical damping term in Eq. (2)
f = gyroscopic damping term in Eq. (2)
g = classical frequency term in Eq. (2)
k = ablation induced frequency term
m = vehicle mass
P, P_{crit} = roll rate, critical roll rate
q_∞ = dynamic pressure
R_1, R_2 = low, high-frequency component magnitudes
S = reference area
s = complex root of transcendental characteristic equation
t = time
V = velocity
α, β = pitch, yaw components of angle of attack
ε = relative strength of ablative pitching moment $\equiv C_{m_{a_\alpha}}/C_{m_\alpha}$
η = ratio of ablation induced to classical frequency terms
ξ = complex angle of attack $= \beta + i\alpha$
λ = real part of characteristic equation root
τ = time lag of ablation induced moment
ω = imaginary part of characteristic equation root
ω_0 = classical nonrolling body frequency
$\Delta\omega$ = component of frequency resulting from roll

Subscripts

0 = normalized by the quantity $g^{1/2}$
1 = characteristic of low-frequency component
2 = characteristic of high-frequency component

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Introduction

RECENT studies[1,2] have shown that the dynamic stability of spinning re-entry vehicles may be grossly different than that resulting from the nominally predicted aerodynamic damping derivative C_{m_q}. In some cases,[2] the effects are so great as to completely override the classical C_{m_q} effect. Possible causes of this behavior may be aerodynamic or mass asymmetries or ablation of the vehicle heat shield. Injection of the ablative material into the boundary layer may result in forces and moments in addition to those normally present. Because of the finite diffusivity and thermal conductivity of this material, the rate of mass injection will lag behind the heat input, and so the forces and moments produced through displacement of the boundary layer may be out of phase with the vehicle motion, and hence may affect the vehicle stability. For spinning vehicles, these out of phase moments may result in both dynamic (C_{m_q}) and Magnus like ($C_{n_{p_\alpha}}$) effects.

Although in some cases anomalous values of the stability derivatives C_{m_q} and $C_{n_{p_\alpha}}$ can be explained by assuming that the phase lags are small enough so that small angle approximations apply, in others[2] trends are noted which are exactly opposite to those predicted by the small angle theory, but which might possibly be explained if larger phase lags were postulated. The present work is concerned primarily with the effects of such large phase lags, and exact solutions to the governing equations are presented for the special case of constant dynamic pressure for a range of pertinent aerodynamic parameters. It is expected that the results obtained from this analysis will shed some light on the case in which dynamic pressure changes occur slowly relative to the angle of attack motions. The case in which the dynamic pressure variations occur on a time scale comparable to the angle of attack motions will be the subject of a sequel to the present paper. The governing equation is of the form of a second-order difference-differential equation with complex constant coefficients. It becomes necessary to solve the related transcendental characteristic equation which arises from the inclusion of the term with retarded argument. The results presented may be used directly in assessing the effects of ablative time lags on vehicle stability and also provide a basis for comparison with approximate techniques of analysis.

In the following, the dynamic model adopted is based upon the assumption that the aerodynamic effects of interest may be described in terms of a lagging pitch moment derivative $C_{m_{a_\alpha}}$ and some associated time lag τ.[1,3]

Dynamic Model

The basic equation of motion in body axes for a spinning re-entry vehicle as derived by Nelson[4] is modified to include the effect of a lagging pitching moment coefficient $C_{m_{a_\alpha}}$ and associated time lag τ. This is done by writing the static pitching moment coefficient C_m as

$$C_m = C_{m_\alpha}\alpha + C_{m_{a_\alpha}}\alpha(t-\tau) \tag{1}$$

where α is the pitch component of the complex angle-of-attack $\xi = \beta + i\alpha$ and $\alpha(t-\tau)$ is the angle of attack occurring τ seconds prior to the current time t. The equation of motion may now be written as

$$\ddot{\xi} + (e + if)\dot{\xi} + g\xi + k\xi(t-\tau) = 0 \tag{2}$$

where some small terms have been dropped, and

$$e = (q_\infty S/V)[C_{N_\alpha}/m - d^2C_{m_q}/2B]$$

$$f = P(2 - A/B), \qquad k = -(q_\infty Sd/B)C_{m_{a_\alpha}}$$

and

$$g = -q_\infty SdC_{m_\alpha}/B - P^2(1 - A/B)$$

It is noted that if $k = 0$, the (classical) solution to Eq. (2) can be written as a superposition of two damped sinusoids

$$\xi = R_1e^{(\lambda_1 + i\omega_1)t} + R_2e^{(\lambda_2 + i\omega_2)t} \tag{3}$$

where the frequencies ω_1 and ω_2 are given approximately as

$$\begin{aligned} \omega_1 &\approx -f/2 + \tfrac{1}{2}(f^2 + 4g)^{1/2} \equiv -\Delta\omega + \omega_0 \\ \omega_2 &\approx -f/2 - \tfrac{1}{2}(f^2 + 4g)^{1/2} \equiv -\Delta\omega - \omega_0 \end{aligned} \tag{4}$$

For most missiles the damping terms λ_1 and λ_2 are nearly equal and are given approximately by

$$\lambda_1 \approx \lambda_2 \approx -e/2$$

Thus when $\Delta\omega > \omega_0$, in body axes the vectors R_1 and R_2 rotate in the same direction, whereas if $\Delta\omega < \omega_0$, they rotate in opposite directions. The roll rate for which the slow frequency component R_1 remains stationary ($\Delta\omega = \omega_0$) is termed "critical."

In order to obtain solutions of Eq. (2), we assume a solution of the form

$$\xi = Re^{st}; \; s = (\lambda + i\omega) \tag{5}$$

where λ and ω are the damping and frequency terms as before. Substituting Eq. (5) into Eq. (2) yields the characteristic equation

$$s^2 + (e + if)s + g + ke^{-s\tau} = 0 \tag{6}$$

Transcendental characteristic equations of the form of Eq. 6 with real coefficients have been studied in the past in connection with control systems[5] containing lags where in general there are an infinite number of roots. General conditions for the stability of that class of system have been given by Pontryagin.[6] It is a peculiarity of dissipative, rotating dynamic systems which leads to the introduction of a complex damping coefficient in which the real part e is associated with the dissipation and the imaginary part f the "gyroscopic damping," is not dissipative. An important consequence of this difference between the form of Eq. (6) appearing here and that studied in relation to other physical problems is the fact that under conditions to be discussed below, Eq. 6 may have only two roots. Thus, the introduction of the lagging moment term need not have as profound an effect on the character of the solution as would normally be expected. Moreover, the determination of the conditions under which there are only two roots to Eq. 6 defines the circumstances under which various approximate solutions of this problem in which effective frequencies are calculated are valid. When only two roots appear, the motion can be expressed in the classical form, Eq. 3. When this is the case, the effects of the ablative pitching moment coefficient may be viewed in terms of an effective C_{m_q} and Magnus term $C_{n_{p_\alpha}}$.

Determination of Regions of Classical Motion

The characteristic equation to be solved (Eq. 6) is a concrete example of the type treated by Pontryagin,[6] and its real and imaginary parts are given by

$$\lambda^2 - \omega^2 + e\lambda - f\omega + g + ke^{-\lambda\tau}\cos\omega\tau = 0 \tag{7}$$

$$2\lambda\omega + f\lambda + e\omega - ke^{-\lambda\tau}\sin\omega\tau = 0 \tag{8}$$

For typical values of the pertinent parameters, the solutions of Eq. (7) and (8) are presented in Fig. 1 and the solution of Eq. (6) is then given by their intersection. A sufficient condition for the existence of only two roots of Eq. (6) is that there be less than a quarter cycle of the function $ke^{-\lambda\tau}\cos\omega\tau$ in the interval $(\omega_b - \omega_a)$, where ω_b and ω_a are the points at which the parabola $-\lambda^2 + \omega^2 - e\lambda + f\omega - g$ (Eq. 7) attains the values $\pm|k|e^{-\lambda\tau}$. This condition can be restated as $(\omega_b - \omega_a)\tau < \pi/2$. Solving for ω_a and ω_b from Eq. (7) and normalizing the result by $g^{1/2}$ yields the condition to be satisfied for the existence of two roots

$$\left[\left(\frac{f^2}{4g} + \frac{e\lambda}{g} + \frac{\lambda^2}{g} + 1 + |\eta|e^{-\lambda\tau}\right)^{1/2} - \left(\frac{f^2}{4g} + \frac{e\lambda}{g} + \frac{\lambda^2}{g} + 1 - |\eta|e^{-\lambda\tau}\right)^{1/2}\right] g^{1/2}\tau < \frac{\pi}{2} \tag{9}$$

Since the expression is transcendental and algebraically complex, it can be reduced to a simpler form only if some assumptions are made regarding the magnitude of certain of its terms. In most cases the quantities $(e\lambda/g)$ and (λ^2/g) are much smaller than unity, so it is not restrictive to reduce relation (9) to the form

$$g^{1/2}\tau < \frac{\pi}{2} \times \left[\frac{1}{(f^2/4g + 1 + |\eta|e^{-\lambda\tau})^{1/2} - (f^2/4g + 1 - |\eta|e^{-\lambda\tau})^{1/2}}\right] \tag{10}$$

Further reduction is made only if we assume that the roll rate f is small compared to $g^{1/2}$ and that $|\eta|e^{-\lambda\tau}$ is small enough ($\ll 1$), so that the term inside the radical can be expanded by the binomial theorem. This step requires that both η and τ have small magnitudes, in which case inequality (10) becomes

$$g^{1/2}\tau < \pi/2|\eta| \tag{11}$$

For example, if $g = 10^3$ and $|\eta| = 0.05$ (typical magnitudes of these quantities), inequality (11) yields a value of τ of one second or less for the existence of a two-component solution.

The ability to represent the solution of Eq. (2) in a classical two component form even for quite large time lags ($\tau \approx 1$ sec) is an important result which, in view of the nature of the characteristic Eq. (6) would not be expected. This result means that standard flight data reduction techniques[4] may be employed to determine the roots λ_1, λ_2, ω_1, and ω_2 directly

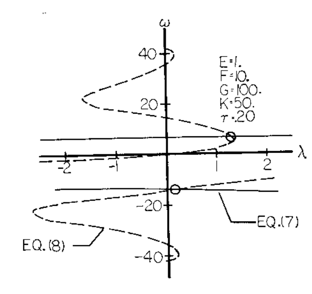

Fig. 1 Determination of characteristics equation roots.

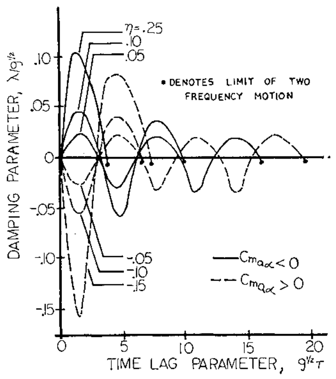

Fig. 2 Damping parameter for zero roll rate.

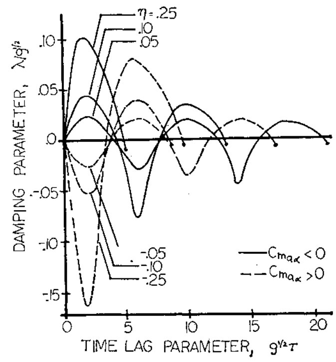

Fig. 4 Low-frequency component damping parameter for $P/P_{crit} \sim 0.25$.

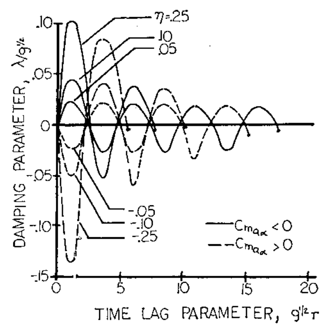

Fig. 5 High-frequency component damping parameter for $P/P_{crit} \sim 0.25$.

from the flight data, so that combinations of $C_{m_{a\alpha}}$ and τ which would account for the observed motion could then be determined.

Presentation of Results

Solutions of Eq. (6) have been generated for time lags for which the implied motion is of the classical form. This was done by first determining the roots for $\tau = 0$ and then considering λ and ω as functions of τ for given values of $e, f, g,$ and k. The subsequent behavior of λ and ω was calculated with a fourth-order Runge-Kutta procedure, utilizing the formulas for the derivatives of λ and ω with respect to τ as obtained from Eqs. (7) and (8). Results are presented for specified cases of roll rate (f) and zero dynamic damping ($e = 0$) for a range of ablative pitching moment coefficients (η), both statically stabilizing ($C_{m_{a\alpha}} < 0$) and statically destabilizing ($C_{m_{a\alpha}} > 0$). The effect of dynamic damping is considered in a subsequent section. The solutions are normalized by $g^{1/2}$ so that the quantities ($\lambda/g^{1/2}$) and ($\omega/g^{1/2}$) are presented as functions of $g^{1/2}\tau$.

Shown in Fig. 2 are the damping parameter histories for the case of zero roll rate (one-component motion). It is apparent from Fig. 2 that the smaller the value of ablative pitching moment (η), the larger the phase lags $g^{1/2}\tau$ which will yield classical motion. It is also seen that λ, the real part of s, is nearly a sinusoidal function of $g^{1/2}\tau$ in the region in which classical motion is predicted. Moreover, the maximum values of λ increase fairly linearly with η. These trends are predicted by the approximate theory of Waterfall.[1] For phase lags of less than about 180°, a statistically stabilizing ablative pitching moment ($C_{m_{a\alpha}} < 0, \eta > 0$) results in a dynamic instability. An interesting feature noted is that the value of $g^{1/2}\tau$ at which multiple (i.e., more than two) roots occur depends on the sign of the term η, as well as its magnitude. This result is explained by referring to inequality (10) and noting the effect of λ on the maximum allowed $g^{1/2}\tau$, i.e., a positive λ decreases the effect of η in inequality (10), whereas a negative λ acts to increase the effect of η.

The variations of body frequency with phase lag are shown for zero roll rate in Fig. 3. The important feature here is that it is possible for a statically destabilizing ablative pitching moment to increase the body frequency above its values for $C_{m_{a\alpha}} = 0$ if $g^{1/2}\tau$ is in the right range.

Figures 4 and 5 show the effect of roll rate on the damping parameter ($\lambda/g^{1/2}$) for a subcritical value of roll rate, $P \approx 0.25$ P_{crit}. The basic features are as described previously, the only difference in the damping parameter ($\lambda/g^{1/2}$) for the high and low-frequency components being the rapidity with which the sign of ($\lambda/g^{1/2}$) changes with increasing $g^{1/2}\tau$. The resulting frequency histories also are similar to the previous case.

It is noted that the strength of the ablative pitching moment $C_{m_{a\alpha}}$ relative to the nominal C_{m_α}, denoted by $\varepsilon \equiv C_{m_{a\alpha}}/C_{m_\alpha}$ can be determined approximately in terms of η and f_0 ($f_0 \equiv f/g^{1/2}$) as

$$\varepsilon \approx \eta/(1 + f_0{}^2/4) \tag{12}$$

Thus, when f_0 is small, ε and η are very close in magnitude, whereas if f_0 is large (e.g., near roll resonance), Eq. (12) must be utilized to relate η to the actual strength of the ablative pitching moment.

As the roll rate is increased to near the critical value (P/P_{crit} ≈ 0.9) the low-frequency component damping term (Fig. 6) changes very slowly, e.g., for $\eta = 0.20$ ($\varepsilon \approx 0.027$) it will be unstable for nearly all values of $g^{1/2}\tau$ shown.

The low-frequency arm damping parameter for roll rate slightly above the critical value ($P/P_{crit} \approx 1.1$) is very nearly identical to that shown in Fig. 6, except that in passing above roll resonance, the effects of positive and negative values of $C_{m_{a\alpha}}$ are reversed from the case previously discussed, i.e., the signs of ($\lambda/g^{1/2}$) are reversed. A statically stabilizing ablative pitching moment here results in a stable damping parameter, while the opposite is true for $C_{m_{a\alpha}} > 0$. As expected, no such trend (i.e., a reversal in the sign of λ) is observed in the high-frequency component, since in passing through

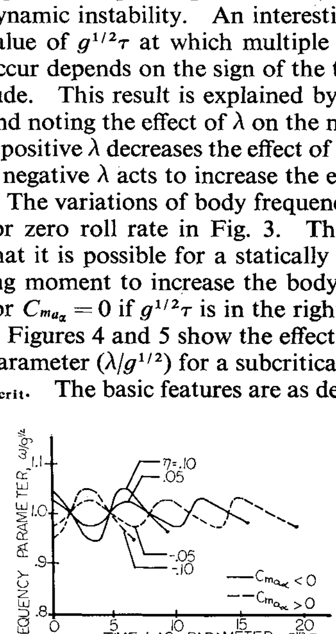

Fig. 3 Frequency parameter for zero roll rate.

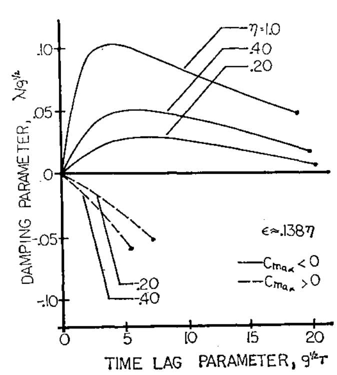

Fig. 6 Low-frequency component damping parameter for $P/P_{crit} \sim 0.9$.

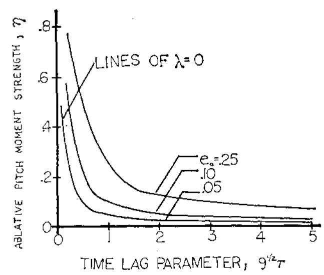

Fig. 7 Low-frequency component stability boundaries for $P/P_{crit} \sim 0.9$.

transient roll resonance the slow frequency component reverses its direction of rotation, while the high-frequency arm is unaffected.

Since the aforementioned cases were calculated for zero dynamic damping ($e_0 \equiv e/g^{1/2} = 0$) we now demonstrate the effect of the inherent vehicle damping e_0 through the use of stability boundary charts, as in Fig. 7. The curves shown give the ablative pitch moment parameter η and time lag τ required to neutralize or overcome the stabilizing effect of e_0 ($e_0 > 0 \Rightarrow$ stable). It is observed from Fig. 7 that the various curves are nearly scaled up from one another by the ratio of the values of e_0. This indicates that the effect of a classically stabilizing C_{m_q} is merely to shift the origin of the damping parameter curve ($\lambda/g^{1/2}$), even for large values of e_0 (e.g., $e_0 = 0.25$, which corresponds to a very large stabilizing C_{m_q}).

Comparison with Approximate Techniques

The results presented herein are compared to those derived from the approximate method due to Waterfall[1] for the cases $\eta = 0.05$ and 0.25, $f_0 = 0$ (Fig. 8), and $\eta = \pm 0.20$, $f_0 = 5$ (Fig. 9) for the low-frequency arm of the motion. In Ref. 1 the effect of a lagging ablative pitching moment is approximated by assuming that $\alpha(t-\tau)$ may be determined from the undamped classical solution given by Eq. (3)[1] with the frequency ω_0 as modified by ablation effects. The resulting damping terms predicted for the high (sub 2) and low (sub 1) frequency modes of the motion are

$$\begin{aligned} \Delta(\lambda_2/g^{1/2}) &= -\eta \sin(\omega_0 + P)\tau(g^{1/2}/2\omega_0) \\ \Delta(\lambda_1/g^{1/2}) &= -\eta \sin(\omega_0 - P)\tau(g^{1/2}/2\omega_0) \end{aligned} \tag{13}$$

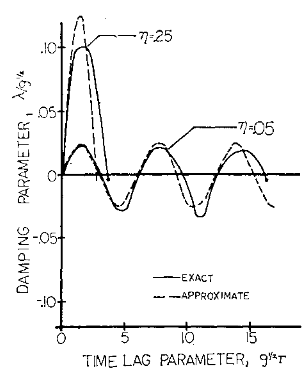

Fig. 8 Comparison of exact and approximate techniques for zero roll rate.

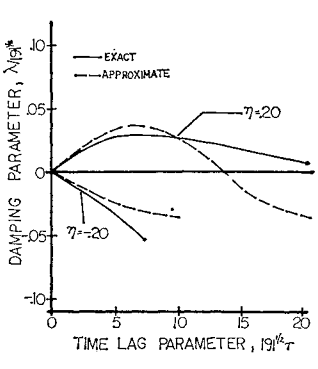

Fig. 9 Comparison of exact and approximate techniques for $P/P_{crit} \sim 0.9$.

where the Δ's indicate that the effects of the lagging pitching moment are to be merely impressed upon the original solution, i.e., the effect can be uncoupled from the classical ones.

In Fig. 8 agreement between exact and approximate solutions is seen to be good for very small values of $g^{1/2}\tau$ for values of η of both 0.05 and 0.25. For the case of small η (0.05) this agreement does not break down until a full cycle in $g^{1/2}\tau$ is passed. For $\eta = 0.25$, however, the approximate and exact solutions differ greatly for low values of $g^{1/2}\tau$. Figure 9 again shows good initial agreement between approximation and exact solution, but the agreement for $g^{1/2}\tau$ greater than about 5 is good in trend but not in level.

It is apparent from Figs. 8 and 9 that for small values of $g^{1/2}\tau$ and η, the theory of Waterfall provides an adequate measure of ablation induced dynamic damping. As η and/or $g^{1/2}\tau$ become larger, however, the agreement tends to break down.

The utility of results obtained from the approximate technique will therefore depend upon the regimes in η and $g^{1/2}\tau$ in which the ablation process falls. At present, for most ablative heat shields, these regimes must be regarded as unknown, so that care must be taken in applying the approximate technique as described here. The conclusions must still be qualified by the neglect of time varying dynamic pressure.

Conclusions

The effects of ablation on R/V dynamics have been modeled by using a difference-differential equation to include the effect of a lagging pitching moment, and exact solutions of the resulting equation have been generated. A significant conclusion is that the motion predicted by this model retains its classical form, displaying only two real frequencies for reasonably large phase lags ($g^{1/2}\tau$), depending on the magnitude of the ablation pitching moment derivative (η). This result is in accord with recent flight-test results,[2] where the motion, although of classical form, exhibits stability derivatives which are of opposite sign to those predicted by the small phase lag approximation. The approximate technique of Waterfall is shown to be satisfactory for small values of $g^{1/2}\tau$ and η, but possibly inadequate as the conditions of small phase lags and small ablative pitching moments are violated. The method of generation of exact solution as presented herein can be easily applied to a wide variety of values of the pertinent parameters of the system. An important question to be raised is whether or not the ablation process in flight can be reasonably approximated by the simplified model adopted here, and the pursuit of the question of consideration of the combined effects of dynamic pressure changes "density damping" and lagging pitching moment components is clearly indicated.

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